# Optimum Beam-To-Column Stiffness Ratio for Portal Frames

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Portal frames of the type shown in *Figure 1* are commonly used in industrial, public and residential buildings. They may be fixed or pinned at the column bases, have flat or pitched roofs, and carry uniform and/or concentrated loads.

Typically, design engineers estimate beam and column sizes based on previous experience to provide a starting point for design and get an estimate of the section properties, such as cross-sectional area and moment of inertia. This is followed by structural analysis in order to determine moment, shear and axial force at various critical locations, such as corner points, supports and beam midspan. The next step involves checking the capacity of individual components (i.e. beams and columns) using the assumed section properties. A number of iterations with readjusted section properties may be required until a close match is reached for finalizing the design. This procedure presents some drawbacks, such as:

1) Although distribution of the straining actions in the frame depends on the relative beam-to-column stiffness ratio ($I_b/I_c$), the design engineer does not have much control of this ratio while trying to match the assumed section properties.
2) The design engineer does not have the chance to optimize. For example, the preset section properties may not lead to roughly equal negative and positive moments.
3) The design engineer does not have the ability to control straining actions or frame deformations that have negative effects. For example, local or global buckling may arise under high axial forces, or midspan deflection may not satisfy code or serviceability requirements.

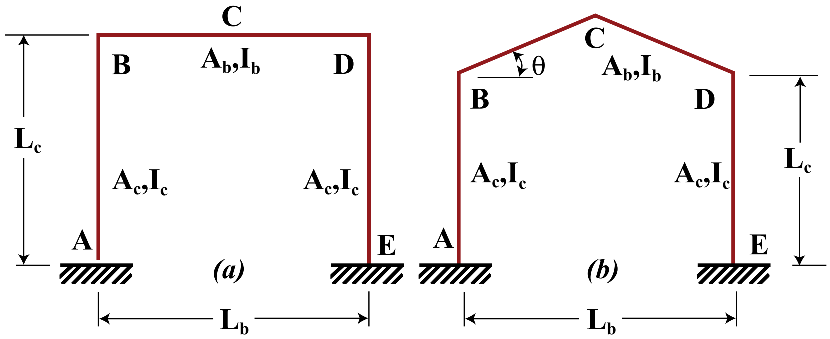


*Figure 1: (a) Flat-portal frame, (b) Pitched–roof-portal frame.*

Overcoming these drawbacks requires understanding the basics of frame behavior and thoroughly studying the effects of beam-to-column stiffness ratio ($I_b/I_c$) on the distribution of straining actions and deformations.

## Basics of Frame Behavior

Consider the flat-roof frame, *ABCDE*, shown in *Figure 2a*, where the frame is fixed at the column ends and the beam is subject to a uniform load. The column deforms vertically and the beam deflects downward, causing its ends to rotate inward. Since the beam is rigidly connected to the column, the column kicks outward if the horizontal movement and rotational restraints of the supports are removed, as shown by the deflected shape *A′B′C′D′E′* in *Figure 2b*. To restore the original conditions of the fixed supports, a horizontal reaction and moment will develop at each support. The horizontal reaction will put the beam in compression, causing its ends to move inward. The support end moment will cause the column to deform outward, as shown by the deflected shape *AB″C″E″D* in *Figure 2c*.

It should be noted that if axial deformations are ignored, which is the case when slope deflection or moment distribution methods are used to analyze the structure, the corner points, that is nodes B and D, do not move vertically under any loading condition, but only rotate and translate horizontally. In the presence of axial deformations, certainly the corner points B and C will move both horizontally and vertically, depending on the relative axial stiffness of the beam and columns.

## Effect on Frame Behavior

As can be seen from the previous discussion, the frame behavior depends on the inward rotation of the corner points B and C. The amount of rotation is controlled by the relative flexural stiffness of the beam and the column; that is, the beam-to-column stiffness ratio. For example, if a small stiffness is assigned to the column, the column does not provide significant rotational resistance to the corner points, and the beam behaves as a simply supported member. This results in high rotation of the corner points and high horizontal reaction forces at the supports. On the other hand, if a large stiffness is assigned to the column, the column provides high rotational resistance to the corner points, and the beam behaves as a fixed-fixed member. This results in small rotations of the corner points and small horizontal reaction forces at the supports.

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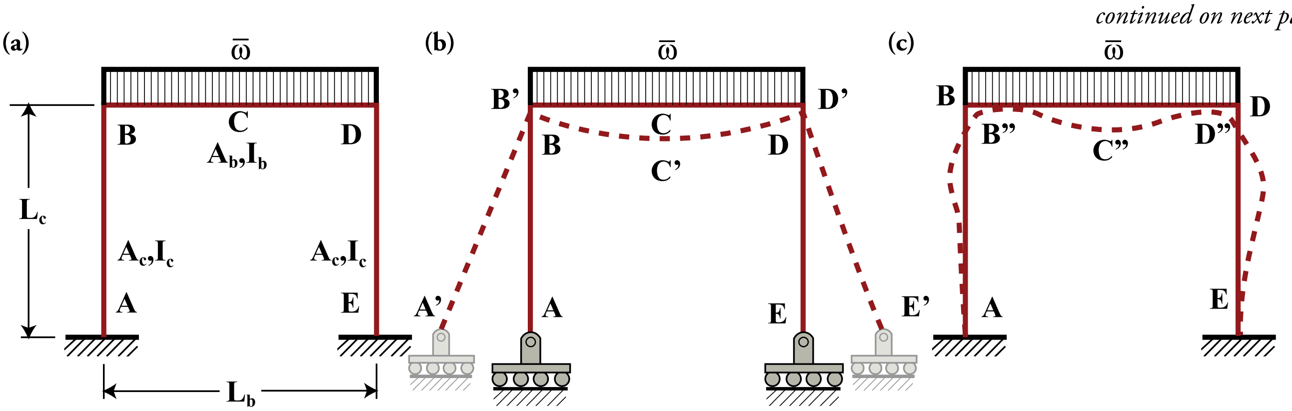



*Figure 2: Basics of frame behavior – (a) Un-deformed shape (b) Released structure (c) Deformed shape*

The following two examples illustrate the effect of the beam-to-column stiffness ratio on the frame behavior. The first is a flat-roof frame, where axial deformations have been ignored and the slope deflection method is used to establish the solution. The second is a pitched-roof frame, where axial deformations are considered and simple finite element routines implemented in Matlab are used to establish the solution.

### *Example 1: Flat-roof frame*

This example considers the frame shown in *Figure 2a*, where the beam is subject to a uniformly distributed load. If axial deformations are ignored, using the slope deflection method and taking advantage of the geometrical and loading symmetry, the following solution can be reached:

$$\textit{Rotation at B}: \alpha_B = +\frac{\omega L_b^2/12}{\left(2+\frac{K_b}{K_c}\right)K_c}$$

$$\textit{Moment at B}: M_B = +\frac{\omega L_b^2/12}{0.5\left(2+\frac{K_b}{K_c}\right)}$$

$$\textit{Moment at A}: M_A = +\frac{\omega L_b^2/12}{\left(2+\frac{K_b}{K_c}\right)}$$

$$\textit{Horizontal reaction at A}: X_A = +\frac{\omega L_b^2/12}{\frac{1}{3}\left(2+\frac{K_b}{K_c}\right)L_c}$$

Where, the ratio $\frac{K_b}{K_c}$ = *beam-to-column stiffness ratio* = $\frac{I_b}{I_c} \times \frac{L_c}{L_b}$, and positive sign in this discussion means clockwise rotation and moment, and inward force for the horizontal reaction.

*Table 1* shows the bending moment diagrams for three selected values of $K_b/K_c$: zero, ⅔ and infinity (∞). $K_b/K_c$ = zero represents a case where the columns are significantly stiffer compared to the beam. In this case, the columns provide full rigidity at the beam ends, and the beam behaves as a pure fixed-fixed member with zero rotation at both ends. This case results in the highest horizontal reaction at the support and axial compression force in the beam.

$K_b/K_c = \infty$ represents a case where the beam is very stiff compared to the columns. In this case, the columns allow the beam ends to rotate, and the beam behaves as a pure simply supported member. Certainly, with a very stiff beam, the beam rotations are very small. Also, this case results in zero horizontal reaction at the support and zero axial force in the beam. These results match those obtained from finite element analysis, as discussed in the next section.

$K_b/K_c$ = ⅔ represents the case where the midspan moment of the beam is equal to the corner moment, which is the case of optimized design for the beam. It should be noted that the value of $K_b/K_c$, required to optimize the beam design, varies with the type of applied loads. For example, if the beam is subject to a single concentrated load *P* at midspan, the optimum is $K_b/K_c$ = zero, and if the beam is subject to a single concentrated load *P* at midspan in addition to the uniformly distributed load ω, such that $P \leq \omega L_b/10$, the optimum is $K_b/K_c$ = ⅔.

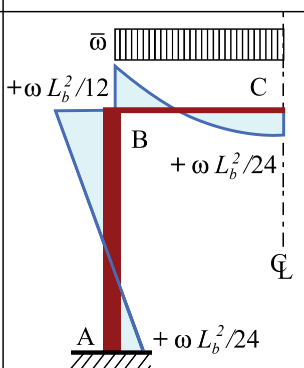
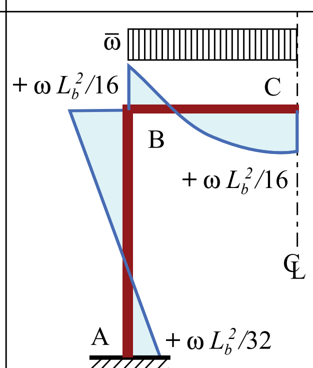
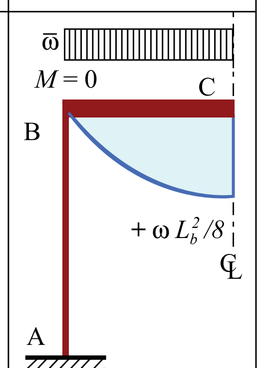
| | **(a)** $K_b/K_c$ = **zero** | **(b)** $K_b/K_c$ = ⅔ | **(c)** $K_b/K_c$ = **infinity** |
|---|---|---|---|
| Bending moment diagram |   |   |   |
| $\alpha_B$ | ≈ zero | $+\frac{\omega L_b^2/12}{\left(\frac{8}{6}\right)K_c}$ | ≈ zero |
| $X_A$ | $+\frac{\omega L_b^2/12}{\frac{2}{3}L_c}$ | $+\frac{\omega L_b^2/12}{\frac{8}{9}L_c}$ | zero |

*Table 1: Bending moment, $\alpha_B$, and $X_A$ for varying values of $K_b/K_c$.*

### *Example 2: Pitched-roof frame*

In this example, the pitched-roof frame shown in *Figure 1b* is analyzed considering variations in beam-to-column moment of inertia ratios, beam-length-to-column-height ratios, roof-pitch at different angles, fixed versus pinned support conditions, and uniformly distributed versus concentrated point loads. Simple finite element routines implemented in Matlab established the solutions for this example. These analyses considered axial deformation effects.

*Figures 3* and *4* show the normalized bending moment versus the beam-to-column stiffness ratio for different pitches due to uniformly distributed loads applied on the beam and concentrated loads applied at the peak point of the frame, respectively. Both figures provide results for fixed and pinned supports and correspond to beam-length-to-column-height ratios equal to 1.5, as shown in *Figure 1b*. The moment values were normalized with respect to $\omega L_b^2$ and $PL_b$ for the uniform load and the concentrated load cases, respectively. Analyses for other beam-length-to-column-height ratios were also undertaken, but for brevity they are not shown here. The solid lines correspond to the analyses for the moment at the corners, i.e. node B, and the dashed line are for the beam mid span, i.e. node C.

Results presented in these figures clearly show that for infinitely stiff columns, i.e. $I_b/I_c$= zero, the bending moment at the beam corners or midspan – indicated in these figures as nodes B and C, respectively – are simply those of fixed-fixed members under either uniformly distributed or concentrated loads. As the beam-to-column stiffness ratio increases, there is a shift in response, and the computed normalized moments become those of simply supported members. These results are consistent with the frame behavior and the results of Example 1, as discussed earlier. Observation of these figures also shows that the bending moment decreases rapidly at the corners and, clearly, increases at mid-span or node C. Although this trend was observed for either the uniformly distributed or concentrated loads, the intersection of these two bending moment curves only occurs in the case of the uniformly distributed loads. This may be an important result in the design of portal frames, as discussed next.

Observation of these figures brings to light some new concepts. For instance, observations of *Figures 3a* and *3b* clearly show that design of a portal frame under uniform loads can lead to a set of optimum design approaches, such as balancing the moment at the corners and midspan. This occurs when the solid lines and the dashed lines intersect for each slope. Certainly other ratios may be considered in design of portal frames under uniformly distributed loads. On the other

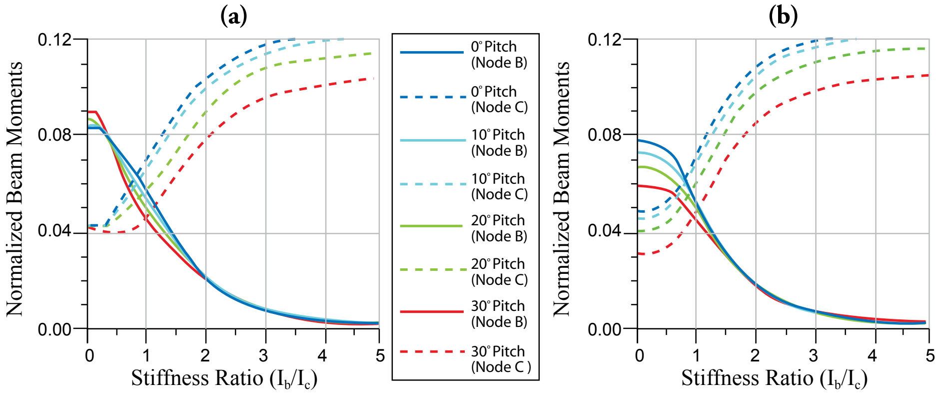



*Figure 3: Results of analysis for uniform loads – (a) $L_b/L_c = 1.5$, Fixed support (b) $L_b/L_c = 1.5$, Pinned support.*

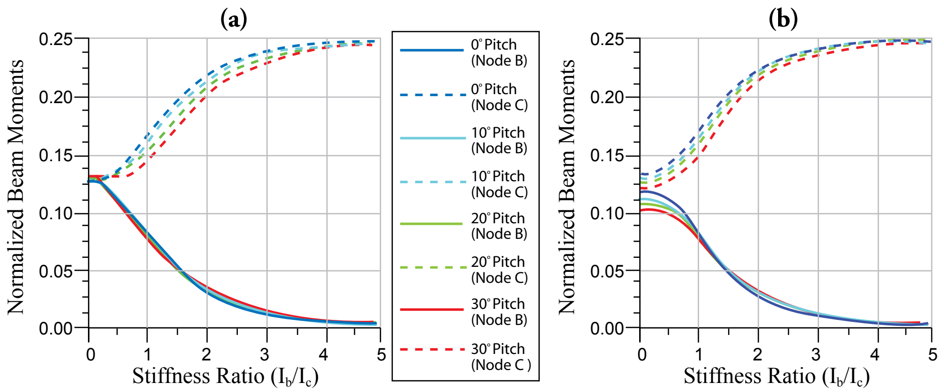



*Figure 4: Results of analysis for point loads – (a) $L_b/L_c = 1.5$, Fixed support (b) $L_b/L_c = 1.5$, Pinned support.*

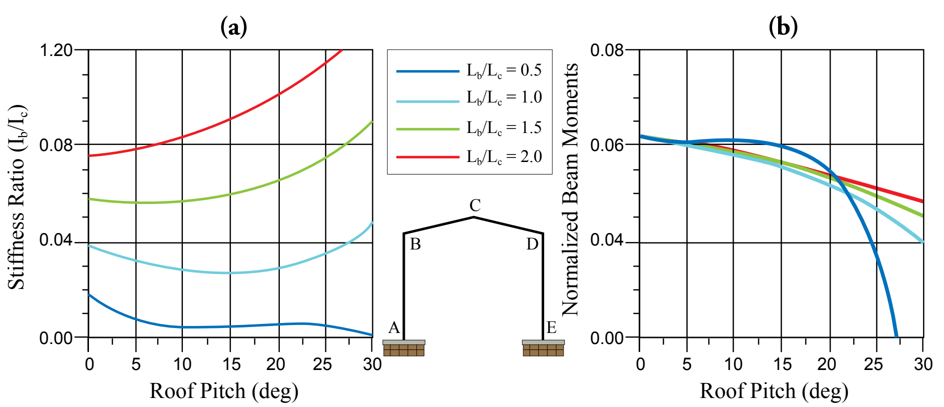



*Figure 5: Optimum design charts for uniform loads – Fixed support frame – (a) Stiffness ratio (b) Normalized bending moments.*

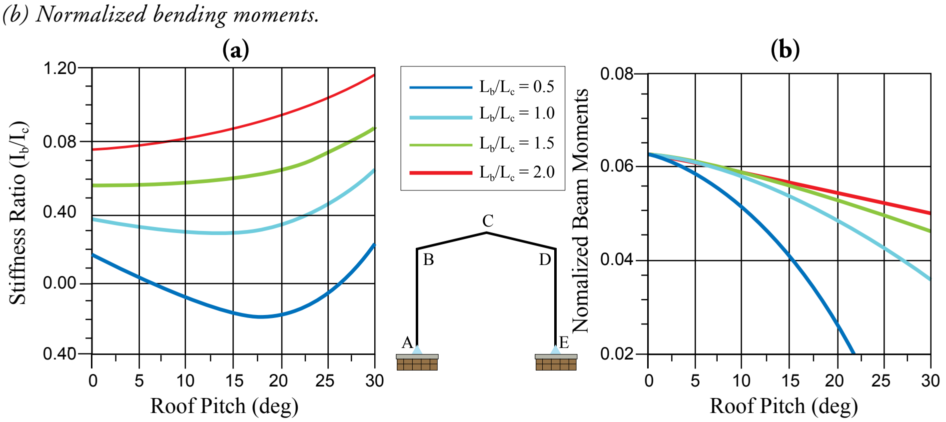



*Figure 6: Optimum design charts for uniform loads – Pinned support frame – (a) Stiffness ratio (b) Normalized bending moments.*

hand, under concentrated loads it appears that no single rule leads to an optimum design. This is consistent with the results discussed from the previous example. These curves also show that the slope of the roof and the support conditions have insignificant effect on the $I_b/I_c$ ratio required to balance the moment at the corners and midspan.

An immediate application of these results is the development of design charts that can be used to optimize the design of portal frames. Examples of such charts are shown in *Figures 5* and *6*. According to *Figures 5a* and *6a*, the designer selects a beam-to-column stiffness ratio that will lead to equal design moments at both ends of the beam. Then, from *Figures 5b* and *6b*, the designer selects the appropriate design moment.

Further examination of *Figures 5* and *6* also shows that:

1) The optimum ratio $I_b/I_c$ is significantly influenced by the roof pitch and the $L_b/L_c$ ratio.
2) For $L_b/L_c$ ratios ≥ 1.0, the beam moments remains almost the same.
3) For $L_b/L_c$ ratios < 1.0, the beam moments are very sensitive to the roof pitch.

## Future Investigations

The design charts developed herein considered the simple rule of matching the same design moments at the corners and at the beam midspan. This does not necessarily indicate that this is the optimum design solution from a cost or performance point of view, but it does serve as a mean to depict how simple computer programs may be used in expediting the design of structures. As shown herein, solution to these problems requires an understanding of the basics of frame behavior and the effects of beam-to-column stiffness ($I_b/I_c$) ratio on the distribution of the straining actions and deformations.

As such, the important point for the engineer to extract from this simple exercise is the definition of an optimum criteria and how to incorporate it into design using simple finite element programs. Certainly, the investigations presented in Examples 1 and 2 can be extended to cover the design of portal frames subject to lateral loads such as wind and earthquake forces, as well as more complex structures such as multi-bay, multi-story frames, and frames where beams and columns have variable moments of inertia (e.g. tapered plate girders).▪

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